An Exercise by Diestel and Dirac's Theorem

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In Graph Theory by Diestel, the fifth edition, on page 30, there is the following exercise.

Exercise. *Show that every connected graph G contains a path or cycle of length at least* $\min\{2\delta(G), |G|\}$.

In Diestel's notation, the minimum degree of the graph G is denoted $\delta(G)$, the number of vertices of G is denoted $|G|$, and the length of a path or cycle is the number of edges it contains.

Before going to the proof, we should look at the statement a bit more closely. In particular, if $|G| \leq 2\delta(G)$, then we would have a path or cycle of length $|G|$. Since the length of a path is one less than the number of vertices, we must have a cycle of length $|G|$ in G. In other words, a Hamiltonian cycle. By this discussion, the exercise implies Dirac's theorem.

Theorem 1. *(G. A. Dirac) A graph with n vertices ($n > 2$) is Hamiltonian if every vertex has degree at least $n/2$.*

Restating the theorem in terms of minimum degree displays the similarity clearly.

Theorem 2. *(G. A. Dirac) A graph with n vertices ($n > 2$) is Hamiltonian if $2\delta(G) \geq n$.*

The condition on Dirac's Theorem also shows us that Diestel did not think about the case where the graph was a single edge.

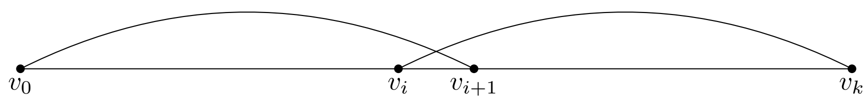
Now we provide the proof for the exercise.

Proof. Let P be the longest path in a connected graph G and let P be indicated by the vertex sequence $v_0, \ldots, v_k$. Suppose that there exists no cycle of length $\min\{2\delta(G), |G|\}$, which implies that $k < 2\delta(G)$.

Consider now the neighbours of v_0. All such neighbours must be on the path P, as otherwise we could extend the path. The same holds true for v_k. Since there are $\delta(G)$ neighbours of each and there only $k < \delta(G)$ such choices for neighbours, there must be i such that v_i is a neighbour of v_k, and v_{i+1} is a neighbour for v_0. This means that there is a cycle C of length $k + 1$, namely the one indicated by the vertex sequence $v_0, \ldots, v_i, v_k, v_{k-1}, \ldots, v_{i+1}, v_0$.

If C is a Hamiltonian cycle, then we have found a cycle of length $|G|$, and we are done.

If the cycle C is not a Hamiltonian cycle, then there exists some vertex v that is a neighbour to one of the vertices of the cycle, as the graph is connected. We can then add the edge to v and delete one edge in the cycle C to obtain a path of size $k+1$, which contradicts the choice of P. □



A depiction of the important construction of the proof is provided above.